

**“ Μένουμε Σπίτι ”**  
**...και κάνουμε Μαθηματικά !**  
**4ο Φύλλο Εργασίας**  
**Αξιοσημείωτες Ταυτότητες**

Καθηγητής: Νικόλαος Δ. Κατσιπης

Άσκηση:

Δίνεται ότι:

$$x = 3 - \sqrt{2} \quad \text{και} \quad y = 3 + \sqrt{2}.$$

Να υπολογίσετε τις τιμές των παραστάσεων:

$$\begin{array}{lll} (\alpha') A = x \cdot y, & (\gamma') \Gamma = x^2 - y^2, & (\epsilon') E = x^3 - y^3, \\ (\beta') B = x^2 + y^2, & (\delta') \Delta = x^3 + y^3, & (\zeta') Z = x^4 - y^4. \end{array}$$

Λύση:

(α')

$$\begin{aligned} A = x \cdot y &= (3 - \sqrt{2})(3 + \sqrt{2}) \\ &= 3^2 - (\sqrt{2})^2 \\ &= 9 - 2 \\ &= 7 \end{aligned}$$

(β')

$$\begin{aligned} B = x^2 + y^2 &= (3 - \sqrt{2})^2 + (3 + \sqrt{2})^2 \\ &= 3^2 - 2 \cdot 3 \cdot \sqrt{2} + (\sqrt{2})^2 + 3^2 + 2 \cdot 3 \cdot \sqrt{2} + (\sqrt{2})^2 \\ &= 9 - 6\sqrt{2} + 2 + 9 + 6\sqrt{2} + 2 \\ &= 22 \end{aligned}$$

(γ')

$$\begin{aligned}
 \Gamma &= x^2 - y^2 = (3 - \sqrt{2})^2 - (3 + \sqrt{2})^2 \\
 &= 3^2 - 2 \cdot 3 \cdot \sqrt{2} + (\sqrt{2})^2 - (3^2 + 2 \cdot 3 \cdot \sqrt{2} + (\sqrt{2})^2) \\
 &= 9 - 6\sqrt{2} + 2 - (9 + 6\sqrt{2} + 2) \\
 &= 9 - 6\sqrt{2} + 2 - 9 - 6\sqrt{2} - 2 \\
 &= -6\sqrt{2} - 6\sqrt{2} \\
 &= -12\sqrt{2}
 \end{aligned}$$

(δ')

$$\begin{aligned}
 \Delta &= x^3 + y^3 = (3 - \sqrt{2})^3 + (3 + \sqrt{2})^3 \\
 &= 3^3 - 3 \cdot 3^2 \cdot \sqrt{2} + 3 \cdot 3 \cdot (\sqrt{2})^2 - (\sqrt{2})^3 + 3^3 + 3 \cdot 3^2 \cdot \sqrt{2} + 3 \cdot 3 \cdot (\sqrt{2})^2 + (\sqrt{2})^3 \\
 &= 27 - 3 \cdot 9\sqrt{2} + 3 \cdot 3 \cdot 2 - 2\sqrt{2} + 27 + 3 \cdot 9\sqrt{2} + 3 \cdot 3 \cdot 2 + 2\sqrt{2} \\
 &= 27 - 27\sqrt{2} + 18 - 2\sqrt{2} + 27 + 27\sqrt{2} + 18 + 2\sqrt{2} \\
 &= 90
 \end{aligned}$$

**Παρατήρηση:** Ισχύει ότι:

$$(\sqrt{2})^3 = (\sqrt{2})^2 \cdot \sqrt{2} = 2\sqrt{2}.$$

(ε')

$$\begin{aligned}
 \text{E} &= x^3 - y^3 = (3 - \sqrt{2})^3 - (3 + \sqrt{2})^3 \\
 &= 3^3 - 3 \cdot 3^2 \cdot \sqrt{2} + 3 \cdot 3 \cdot (\sqrt{2})^2 - (\sqrt{2})^3 - (3^3 + 3 \cdot 3^2 \cdot \sqrt{2} + 3 \cdot 3 \cdot (\sqrt{2})^2 + (\sqrt{2})^3) \\
 &= 27 - 3 \cdot 9\sqrt{2} + 3 \cdot 3 \cdot 2 - 2\sqrt{2} - (27 + 3 \cdot 9\sqrt{2} + 3 \cdot 3 \cdot 2 + 2\sqrt{2}) \\
 &= 27 - 27\sqrt{2} + 18 - 2\sqrt{2} - 27 - 27\sqrt{2} - 18 - 2\sqrt{2} \\
 &= -58\sqrt{2}
 \end{aligned}$$

(ς')

$$\begin{aligned}
 Z &= x^4 - y^4 = (x^2)^2 - (y^2)^2 \\
 &= (x^2 - y^2) \cdot (x^2 + y^2) \\
 &= \Gamma \cdot \text{B} \\
 &= -12\sqrt{2} \cdot 22 \\
 &= -264\sqrt{2}
 \end{aligned}$$

*“Τα Μαθηματικά δεν γνωρίζουν φυλές ή γεωγραφικά όρια, για τα μαθηματικά ο πολιτισμός όλου του κόσμου είναι μία χώρα.”*

David Hilbert, 1862-1943, Γερμανός μαθηματικός.